Pure and Applied Mathematics Quarterly
Volume 19, Number 1, 233–265, 2023

Congruences via fibered motives

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Introduction

An archetypal congruence between automorphic forms, and the first one occurring in level 1, is Ramanujan's $\tau(n) = \sigma_{11}(n) \mod 691$. It can be proved as follows. There exists a non-zero modular form of weight 12 with integer coefficients and constant term 1: take E_4^3. As the space of modular forms of weight 12 is 2–dimensional, there must be a linear relation of the shape $-\frac{B_{12}}{24}E_4^3 = -\frac{B_{12}}{24}E_{12} + m\Delta$ with some rational m (which happens to be 600/91, but the exact value is not essential for the purpose of the argument). Since $B_{12} = 0 \mod 691, m$ must be $-1 \mod 691$ by comparing the first expansion coefficients, so reducing mod 691 we have $0 = \sum_{n=1}^{\infty}(\sigma_{11}(n) - \tau(n))q^n \mod 691$.

Miles away from this subject (but see [33, 40]), the number $|S_{23}|$ of homotopy 23–spheres, 69524373504, is divisible by 691. The reason is that one of the two essential factors of $|S_{4k-1}|$, the order of the subgroup S^{bp}_{4k-1} of homotopy spheres bounding parallelizable $4k$–manifolds M, is $2^{2k-2}(2^{2k-1}-1)\cdot$ numerator of $\frac{4B_{2k}}{k}$. If one were to explain the mechanism behind this discovery of Kervaire and Milnor [36, 39] in a telegram, one could say that the only invariant distinguishing such spheres S is the signature $\sigma(M)$, that any multiple of 8 can be realized as a signature, and that by Hirzebruch's signature formula those signatures that arise in M's bounding the standard sphere are all integer multiples of $2^{2k-1}(2^{2k-1}-1)\frac{B_{2k}}{k}$. Could the existence of certain topologies be systematically linked to congruences between automorphic forms with the numerology of Hirzebruch's formula as intermediary?

Mirror duality offers a different mechanism to realize the same principle. C. Poor and D. Yuen [43] prove the following theorem. Denote, in general, the space of paramodular cuspforms of weight k and paramodular level N by $S_k(K(N))$, the space of Jacobi cuspforms of weight k and index N by $J^{\mathrm{cusp}}_{k,N}$ and by Grit the Gritsenko lift

$$\mathrm{Grit} : J^{\mathrm{cusp}}_{k,N} \longrightarrow S_k(K(N)).$$

Received November 19, 2021.

*Vasily Golyshev thanks the Institut des Hautes Études Scientifiques for the extraordinary support it gave him in 2022.

The dimension of $J^{\mathrm{cusp}}_{3,79}$ is 7, while Ibukiyama's formula gives $S_3(K(79)) = 8$. There is a nonlift Hecke eigenform $F_{79} \in S_3(K(79))$ which has integral coefficients. This form is congruent modulo 32 to a Gritsenko lift from $\mathrm{Grit}(J^{\mathrm{cusp}}_{3,79}(\mathbb{Z}))$ and any other such congruence is a reduction of this one. (This mod 32 congruence is of Type II in the terminology we introduce in Section 1.)

Poor and Yuen construct this form as a rational expression in Gritsenko lifts of the so called theta blocks [27], and compute the following Euler factors of the L-function of F_{79}.

$$
\begin{aligned}
Q_2(F_{79}, T) &= 1 + 5T + 14T^2 + 40T^3 + 64T^4 \\
Q_3(F_{79}, T) &= 1 + 5T + 42T^2 + 135T^3 + 729T^4 \\
Q_5(F_{79}, T) &= 1 - 3T + 80T^2 - 375T^3 + 15625T^4
\end{aligned}
$$

Think of the set of all reasonable (automorphic, Hasse–Weil, Selberg–class) L-functions with algebraic coefficients as vertices of a graph, where two vertices are connected by an edge if there is a congruence between them. Koji Doi posed a question of finding paths connecting a given pair of L-functions in this graph.

We conjecture the existence of a different, 'Type III', congruence mod 5. This mod 5 congruence is between F_{79} and a particular Hilbert modular form f_{79} of parallel weight 2 and level $(-1+4\sqrt{5})$ with an L-function of conductor $25 \cdot 79$ with the Euler factors

$$
\begin{aligned}
Q_2(f_{79}, T) &= 1 - T^2 + 4T^4 \\
Q_3(f_{79}, T) &= 1 + 2T^2 + 9T^4 \\
Q_5(f_{79}, T) &= 1 + 2T + 5T^2
\end{aligned}
$$

By this we mean that the respective Euler factors are congruent $\bmod 5$ as polynomials in $T = p^{-s}$ for almost all primes p in $\mathbb{Z}$. Furthermore, we hypothesize that it could be ultimately the existence of a single Fano fourfold, namely the section $G_{2,2}$ of the grassmannian $G(2,5)$ by two quadratic hypersurfaces in its Plücker embedding, and the divisibility of the characteristic number $c_1^4 = 20$ by 4 and 5 that is hidden behind an *infinite family* of such congruences. The reason is that characteristic numbers of Fano varieties determine the integral monodromy of their mirror pencils via the gamma class. The primes l that divide certain characteristic numbers such as c_1^4 can have the property that the monodromy of the mirror pencil with $\mathbb{F}_l$-coefficients has an exceptionally small image. For every fiber of the mirror pencil, this will in turn produce a drop of the residual $\bmod l$ Galois representation arising in its cohomology, cf. [19], and suggest the existence of a mod l-congruent Galois representation, also arising from algebraic geometry. This is pretty much the content of the present paper. Finally, the standard automorphy expectations will predict a congruence between the automorphic forms Langlands–dual to these Galois representations.

Strategy of a proof. At the moment, we see no direct proof of this congruence along the lines of the proofs of the Ramanujan or Poor–Yuen congruences cited above and dealing with the Fourier expansion coefficients directly. Instead, a roundabout strategy — to prove the congruence by an argument in arithmetic geometry rather than automorphic forms — could involve the following steps. First, one would realize both L-functions as Hasse–Weil L-functions. This is possible in principle because both modular forms are of geometric type: the Hilbert modular form f_{79} arises from the cohomology of an elliptic curve over $\mathbb{Q}(\sqrt{5})$, while F_{79} is known by Weissauer [49] to be coming from a piece in the H^3 of the respective paramodular threefold. However, this space is too big exactly because it contains the components that come from the lifts. In an ideal situation one might be able to produce a threefold whose H^3 would have only that rank 4 piece without any extra classes. After this, one might be able to finish off the proof of the congruence by an argument in arithmetic geometry rather than automorphic forms.

It is this argument, the existence of a congruence between two $\mathrm{Gal}(\mathbb{Q})$-representations, one arising in the H^3 of a threefold over $\mathbb{Q}$ and one induced from the H^1 of an elliptic curve over $\mathbb{Q}(\sqrt{5})$, that concludes Section 2 of this paper. In Section 1, we review the necessary background on rank 4 weight 3 Calabi–Yau motives, congruences of Type I, II, and III, mirror symmetry, Landau–Ginzburg models and Picard–Fuchs differential equations therein. We then introduce the motive of a variety $\mathbf{Y}_{79}$ occurring as a fiber in the Landau–Ginzburg model $\mathcal{Y}$ of $G_{2,2}$. We fiber it out over $\mathbb{G}_m(t)$ into a family of motives occurring in the relative H^2 in a family of abelian surfaces of the form $E_L \times_t E_A$, and compute its l-adic realization as $H^1(\bar{\mathbb{P}}^1, \mathcal{L} \otimes_t \mathcal{A})$ where $\mathcal{A}$ and $\mathcal{L}$ are the ($*$-extended) local systems arising in the relative H^1's of the factors. We then notice that $\mathcal{A}$ is a 5–congruence sheaf. Roughly speaking, this means that for any closed point x in a model of $\mathbb{G}_m$ over $\mathbb{Z}$ over which $\mathcal{A}$ is lisse, say with a residue field $\mathbb{F}_q$, we have $\mathrm{Tr}\ \mathrm{Frob}\ _x = 1 + q \pmod 5$. It follows that what the cohomology $H^1(\bar{\mathbb{P}}^1, \mathcal{L} \otimes_t \mathcal{A})$ 'sees mod 5' is essentially the stalks of $\mathcal{L}$ at the ramified points of $\mathcal{A}$. In order to make the argument as low–tech and transparent as possible we indeed run it at the level of trace functions rather than sheaves.

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For the reader's convenience we review briefly by way of comparison paramodular forms, Hilbert modular forms, and their L–functions.

Paramodular forms and the Euler factors of their L–functions. This is borrowed from a formulaire of C. Poor based on [34].

– Space: Siegel upper half space: $\mathcal{H}_n = \{Z \in M^{\mathrm{sym}}_{n\times n}(\mathbb{C}) : \operatorname{Im} Z > 0\}$.

– Group action: symplectic group: $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{R})$ acts on $Z \in \mathcal{H}_n$ by $\sigma \cdot Z = (AZ+B)(CZ+D)^{-1}$.

– Slash: For $F : \mathcal{H}_n \longrightarrow \mathbb{C}$, $\sigma \in \mathrm{Sp}_n(\mathbb{R})$, $(F|_k\sigma)(Z) = \det(CZ+D)^{-k}F(\sigma\cdot Z)$.

– Arithmetic group: Γ is the paramodular group of level N,

$$\Gamma = K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}), \quad * \in \mathbb{Z}.$$

– Modular forms: paramodular forms, or Sieg $M_k(\Gamma)$ is the $\mathbb{C}$–vector space of holomorphic $F : \mathcal{H}_2 \longrightarrow \mathbb{C}$ that are 'bounded at the cusps' and that satisfy $F|_k\sigma = F$ for all $\sigma \in \Gamma$.

– Cusp forms: $S_k(\Gamma) = \{F \in M_k(\Gamma)$ that 'vanish at the cusps'$\}$.

– Hecke operators: averaging over double cosets,

$$T_{0,1}(p) = K(N)\,\mathrm{diag}(p,p,1,1)K(N)$$
$$T_{1,0}(p) = K(N)\,\mathrm{diag}(p,p^2,p,1)K(N).$$

– Euler factors at good primes: let F be a Hecke–eigen newform:

$$F|_kT_{0,1}(p) = \lambda_pF,$$
$$F|_kT_{1,0}(p) = \mu_pF.$$

Then

$$Q_p(F,T) = 1 - \lambda_pT + (p\mu_p + p^{2k-3} + p^{2k-5})T^2 - p^{2k-3}\lambda_pT^3 + p^{4k-6}T^4.$$

Hilbert modular forms with trivial character over real qudratic fields. We follow the conventions of [15]. Let F be a real quadratic field, for simplicity of narrow class number one.

– Space: $\mathcal{H}_1 \times \mathcal{H}_1$.

– Group action: $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ acting on $\mathcal{H}_1 \times \mathcal{H}_1$ componentwise: $\gamma = (\gamma_1, \gamma_2) = \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right)$ acts by

$$\gamma(z_1, z_2) = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2} \right).$$

– Slash: let $k = (k_1, k_2)$ be a pair of integers of the same parity, and put $k_0 = \max(k_1, k_2)$.
Then

$$(f|_k \gamma)(z) = \det(\gamma_1)^{k_0/2+k_1/2-1} \det(\gamma_2)^{k_0/2+k_2/2-1} (c_1 z_1 + d_1)^{-k_1} \times (c_2 z_2 + d_2)^{-k_2} f(\gamma z).$$

– Arithmetic group: consider $G = \mathrm{GL}_2(\mathrm{F})$ as mapped into $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ via the two embeddings of F into $\mathbb{R}$. Let $\mathfrak{n}$ be an ideal in $\mathcal{O}_F$. Define congruence subgroup $\Gamma = \Gamma_0(\mathfrak{n})$ of $G(\mathcal{O}_\mathrm{F})$ as

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{lll} a \in \mathcal{O}_\mathrm{F}, & b \in \mathcal{O}_\mathrm{F}, & \\ c \in \mathfrak{n}, & d \in \mathcal{O}_\mathrm{F}, & ad - bc \in \mathcal{O}_\mathrm{F}^\times \end{array} \right\}.$$

– Cusp forms: S_k is the space of Γ–modular forms that 'vanish at the cusps'.

– Hecke operators: for each prime ideal $\mathfrak{p} = (\pi)$, the Hecke operator $T_\mathfrak{p}$ at $\mathfrak{p}$ is defined by

$$T_\mathfrak{p} = \Gamma_0(\mathfrak{n}) \operatorname{diag}(\pi, 1) \Gamma_0(\mathfrak{n}).$$

– Euler factors at good primes: let f be a Hecke–eigen newform:

$$f|_k T_\mathfrak{p} = \lambda_\mathfrak{p} f,$$

Then

$$Q_\mathfrak{p}(f, T) = 1 - \lambda_\mathfrak{p} T + N(\mathfrak{p})^{k_0 - 1} T^2.$$

1. Congruences for $(1, 1, 1, 1)$–motives

1.1. Calabi–Yau motives

For a Calabi–Yau threefold X defined over $\mathbb{Q}$ with Hodge numbers $h^{3,0} = h^{0,3} = 1, h^{2,1} = h^{1,2} = a$, Poincaré duality defines a non-degenerate alternating form on the third cohomology $H^3 X$, making it a so–called symplectic motive of rank $2+2a$. The Hodge number $a = h^{1,2}$ has an interpretation of the dimension of the local moduli space for X. In the rigid case, $a = 0, H^3X$ is a rank 2 motive of weight 3, and its L–function can be described in terms of classical modular forms. The p–th Euler factor has the form

$$\det(1 - T \cdot \mathrm{Frob}_p\,|_{H^3_{\text{ét}}(\bar{X},\mathbb{Q}_l)}) = 1 - a_p T + p^3 T^2$$

and the number a_p is the p–th Fourier coefficient of a new eigenform in $S_4(\Gamma_0(N))$ with rational coefficients, see [25, 17]. But the correspondence is not as straightforward as in the elliptic curve case: we do not know which weight 4 modular forms can be realized by Calabi–Yau threefolds and furthermore, examples show that there exist topologically distinct rigid Calabi–Yau threefolds with the same modular form. Many examples can be found in the book by C. Meyer [41] but much remains to be done.

Our main interest lies in the case $a = 1$. In this case H^3X is a symplectic motive of rank four, with Hodge numbers $(1, 1, 1, 1)$. Its Euler factors take the form

$$\det(1 - T \cdot \mathrm{Frob}_p\,|_{H^3_{\text{ét}}(\bar{X},\mathbb{Q}_l)}) = 1 + \alpha_p T + \beta_p p T^2 + p^3 \alpha_p T^3 + p^6 T^4.$$

Some of these motives naturally appear as the fibers of one–parameter families of Calabi–Yau threefolds $\mathcal{X} \longrightarrow \mathbb{P}^1$. Many such families are known explicitly, and have been studied in varying degrees of detail. The simplest are the 14 hypergeometric families, which directly generalize the famous Dwork pencil described by P. Candelas, X. de la Ossa, P. Green and L. Parkes [10] and are known from the earliest days of mirror symmetry. For general values of t one obtains irreducible $(1, 1, 1, 1)$–motives. Further playing ground is provided by the AESZ list [1] which contains about 500 so–called *Calabi–Yau differential equations*. These are self–dual differential equations of order 4 that arise as Picard–Fuchs equations in 1–parameter families of Calabi–Yau motives. An effective computational method was put forward in the paper [9] that attaches Euler factors to each non-singular point for such a differential equation.

1.2. The paramodular quest

Apart from the congruence subgroups in $\mathrm{Sp}_4(\mathbb{Z})$, the paramodular groups $K(N)$ of level N that appeared in the introduction have great relevance. By definition, the Siegel modular threefolds

$$Y(N) := \mathcal{H}_2/K(N)$$

have a natural interpretation as moduli spaces for abelian surfaces with $(1:N)$–polarization. The algebraic geometric study of projective models has a long history. A model of $Y(11)$ as a cubic hypersurface in $\mathbb{P}^4$ appears already in Klein's paper [37], cf. [29]; abelian surfaces with $(1:5)$ polarization appear in the study of the Horrocks–Mumford bundle on $\mathbb{P}^5$. The question of unirationality of $Y(N)$ is also addressed by V. Gritsenko [26] and in [29, 30].

A local newform theory for the paramodular levels in GSp(4) was developed by B. Roberts and R. Schmidt in [45]. This theory shows a striking analogy of $K(N)$ with the classical theory for $\Gamma_0(N)$ and suggests that the threefolds $Y(N)$ are natural generalizations of the modular curves $X_0(N)$. The Langlands correspondence predicts that symplectic motives of rank $2n$ come from automorphic forms for the split orthogonal group $\mathrm{SO}(n,n+1)$. In [28] and [47] the local newform theory was developed for paramodular levels in higher–rank orthogonal groups, generalizing the work B. Roberts and R. Schmidt had done for Sp_4, which is isogenous to $\mathrm{SO}(2,3)$.

We will say that a weight n motive is of Calabi–Yau type if $h^{n,0}=1$. One might then ask which $(1,1,1,1)$–motives of Calabi–Yau type are paramodular. More precisely, if M is such a motive of conductor N, does there exist a weight 3 paramodular newform $F \in S_3(K(N))$ such that

$$L(s,M)=L(s,F), \quad F \in S_3(K(N))\,?$$

For instance, C. Poor, J. Shurman, and D. Yuen suggested paramodular forms of levels 525 and 257 whose Euler factors matched with those of hypergeometric motives of these conductors produced by H. Cohen and D. Roberts [12].

One may go further and ask when such a $(1,1,1,1)$–motive of Calabi–Yau type can be realized geometrically inside the cohomology of a smooth projective Calabi–Yau threefold with $b_3=4$.

1.3. Forms of low conductor

In [3], A. Ash, P. Gunnells, and M. McConnell found indications for the existence of cusp forms in H^5 for congruence subgroups $\Gamma_0(N) \subset SL_4(\mathbb{Z})$ for $N=61,73,79$ and conjectured them to be lifts of Siegel modular forms. Using rational combinations of Gritsenko lifts, C. Poor and D. Yuen [43] constructed paramodular cusp forms $F_N \in S_3(K(N))$ for $N=61,73,79$ and computed Euler factors for $p=2,3,5$. Applying the Selberg–Stark method of guessing numerically L–functions as Dirichlet series with integer coefficients under the assumption of analytic continuation and the functional equation $\Lambda(s) = \Lambda(4-s)$ for a putative completed L–function

$$\Lambda(s) = \left(\frac{N}{\pi^4}\right)^{s/2} \Gamma\left(\frac{s-1}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) L(s, F_N),$$

A. Mellit was able to guess the first few hundred coefficients of the L–series for F_N, $N = 61, 73, 79, \ldots$. Subsequent computations of algebraic modular forms by J. Hein [32], W. Ladd [38] and G. Tornaría produced Euler factors up to $p = 31$ and Dirichlet series up to more than 1000 terms, confirming these findings. For further information we refer the reader to [44].

1.4. Congruence types

We will say that two $\mathrm{Gal}(\mathbb{Q})$–representations R_1, R_2 are congruent $\bmod N$ if for almost all primes p we have $Q_p(R_1, T) \equiv Q_p(R_2, T) \pmod N$. Suppose now that we are given a rank 4 weight 3 Calabi–Yau motive M and a congruence $R_M \equiv R \pmod N$ between its Galois representation $R_M = H^3_{\text{ét}}(M, \mathbb{Q}_l)$ and another $\mathrm{Gal}(\mathbb{Q})$–representation R. We will consider the dimensions of the absolutely irreducible pieces occurring in R and will say that the congruence is of

- Type I, if all the pieces are 1–dimensional (i.e. if at the expense of restricting R to a finite–index subgroup it breaks into an extension of four characters);
- Type II, the dimensions are 1, 1, and 2;
- Type III, if the dimensions are 2 and 2.

1.5. Examples

It was shown by P. Candelas, X. de la Ossa and F. Rodriguez Villegas [8] that

$$Q_p \equiv (1-T)(1-pT)(1-p^2T)(1-p^3T) \pmod 5$$

for almost all p and almost all fibers of the Dwork family, so that there exists a congruence of type I of the respective Galois representation with $\mathbb{Q}_l \oplus \mathbb{Q}_l(-1) \oplus \mathbb{Q}_l(-2) \oplus \mathbb{Q}_l(-3)$. (We point out that with our numerical definition only the semisimplification class matters.) This was dubbed arithmetic mirror symmetry, as in the reduction mod 5 we see an arithmetic analogue of a degeneration near a MUM–point. A Type II congruence mod 43 was observed by C. Poor and D. Yuen at the level of automorphic forms for the paramodular form F_{61} of level 61 they constructed; it predicts a Type II congruence for the Galois representation in the H^3 of a certain threefold with bad reduction at 61 that we have identified as a hypersurface in $\mathbb{G}_m^4$, but no direct geometric proof has so far been found. Another Type II congruence, mod 19, for the same form has recently been proved in [20].

For Type III congruences that are not simultaneously Type I or II one might expect each $Q_p(M,T) \mod N$ to factorize into quadratic factors that are irreducible for a positive proportion of primes p. This type of behavior is found in the L–function of F_{79} with the congruence modulus $N=5$, as may be observed in the following table:

p	Q_p	$Q_p \pmod 5$
5	$1-3T+16pT^2-3p^3T^2+p^6T^4$	$1-3T \pmod 5$
7	$1-15T+26pT^2-15p^3T^3+p^6T^4$	$(1+3T+3T^2)(1+2T+3T^2) \pmod 5$
11	$1-26T+122pT^2-26p^3T^3+p^6T^4$	$(1+4T+T^2)(1+2T)(1+3T) \pmod 5$
13	$1+15T+12pT^2+15p^3T^3+p^6T^4$	$(1+3T^2)^2 \pmod 5$
17	$1+60T+134pT^2+60p^3T^3+p^6T^4$	$(1+T+2T^2)(1+4T+2T^2) \pmod 5$
19	$1-32T-350pT^2-32p^3T^3+p^6T^4$	$(1+2T+4T^2)(1+3T)^2 \pmod 5$
23	$1-50T+274pT^2-50p^3T^3+p^6T^4$	$(1+3T+3T^2)(1+2T+3T^2) \pmod 5$

We will introduce a certain Calabi–Yau threefold, $\mathbf{Y}_{79}$, that emerges from mirror symmetry and is a candidate for a geometrical realization of F_{79}. The description of its H^3 as that of a specific threefold hypersurface in $\mathbb{G}_m^4$ defined by a Laurent polynomial, enables us to interpret it as a fibered motive, from which the presence of this 5–congruence of Type III will follow.

2. Apéry and its convolutions

2.1. The Grassmannian $G(2,5)$

We will now illustrate the principle that certain characteristic numbers of Fano manifolds lead to congruences for the motives that appear in the fibers of an associated Landau–Ginzburg model. We will take a brief look at mirror symmetry of the Grassmannian $G=G(2,5)$.

The ample generator h of $\mathrm{Pic}(G)=\mathbb{Z}$ represents the Plücker embedding in $\mathbb{P}^9$ as a Pfaffian 6–fold of degree 5. The (small) quantum cohomology $QH^*(G)$ of G is a certain deformation of the classical cohomology ring $H^*(G)$ defined in terms of counting of rational curves, parameterized by a variable t on the dual torus to $\mathrm{Pic}(G)^\vee$. It gives rise to a connection in a trivial bundle over $\mathbb{G}_m$ with the fiber $H^*(G,\mathbb{C})$ of the form $\Theta\bar{\zeta}=h*\bar{\zeta}$ with $\Theta=t\frac{d}{dt}$ and $\bar{\zeta}$ a basis of constant cohomology–valued sections. Here $h*$ means the $\mathbb{C}[t,t^{-1}]$–linear operator of quantum multiplication by the hyperplane class.

Choosing the fundamental–class section for a cyclic vector, one finds [4] the quantum differential operator of order 10 which annihilates it:

$$Q = \Theta^7(\Theta-1)^3 - t\Theta^3(11\Theta^2+11\Theta+3) - t^2.$$

The exponents 7 and 3 encode the (even) Betti numbers $1,1,2,2,2,1,1$ of G. It has a special power series solution

$$\Psi(t) = \sum_{n=0}^{\infty} \frac{1}{n!^5} A_n t^n = 1 + 3t + \frac{19}{(2!)^5}t^2 + \frac{147}{(3!)^5}t^3 + \frac{1251}{(4!)^5}t^4 + \ldots$$

where

$$A_n = \sum_{k,l} \binom{n}{k}^2 \binom{n}{l}\binom{k}{l} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$$

are the small Apéry numbers, which played a key role in Apéry's irrationality proof of $\zeta(2)$, [2] and satisfy the recursion

$$(n+1)^2 A_{n+1} - (11n^2+11n+3)A_n - n^2 A_{n-1} = 0, \quad n \geq 1.$$

As the canonical class $c_1(G) = -5h$, a complete intersection $G_{d_1,d_2,\ldots,d_r}$ of G by r hypersurfaces of degrees $d_1, d_2, \ldots, d_r$ is Calabi–Yau precisely when $d_1 + d_2 + \ldots + d_r = 5$. The factorially modified series

$$\phi(t) = \sum_{n=0}^{\infty} \frac{(d_1 n)!(d_2 n)!\ldots(d_r n)!}{n!^5} A_n t^n,$$

satisfies a fuchsian differential equation and is called the period function. By Givental's *quantum Lefschetz principle* [11] it is the Picard–Fuchs equation of the corresponding mirror family of Calabi–Yau varieties dual to $G_{d_1,d_2,\ldots,d_r}$. Below we list the corresponding differential operators for the mirrors of the Calabi–Yau sections of G of dimension $1,2,3$.

degrees	dim	operator
$1,1,1,1,1$	1	$\theta^2 - t(11\theta^2+11\theta+3) - t^2(\theta+1)^2$
$2,1,1,1$	2	$\theta^3 - 2t(2\theta+1)(11\theta^2+11\theta+3)$ $-4t^2(2\theta+1)(\theta+1)(2\theta+3)$
$2,2,1$	3	$\theta^4 - 4t(2\theta+1)^2(11\theta^2+11\theta+3) - 16t^2(2\theta+1)^2(2\theta+3)$
$3,1,1$	3	$\theta^4 - 3(3\theta+1)(3\theta+2)(11\theta^2+11\theta+3)$ $-9t^2(3\theta+1)(3\theta+2)(3\theta+4)(3\theta+5)$

All these operators can be seen as convolutions of the Apéry operator with (scaled) hypergeometric operators. It is this convolution structure that has consequences for the arithmetic of the associated motives.

2.2. The Apéry operator

$\mathcal{P}_A := \theta^2 - t(11\theta^2+11\theta+3) - t^2(\theta+1)^2$ has singularities in a set Σ_A consisting of four points: $0, \infty$ and the two roots

$$a_1 := \frac{-11-5\sqrt{5}}{2}, \quad a_2 := \frac{-11+5\sqrt{5}}{2}$$

of the polynomial

$$\Delta_A(t) = 1 - 11t - t^2.$$

We will denote the local system of solutions on the smooth open $U := \mathbb{P}^1 \setminus \Sigma_A$ by $\mathcal{A}_U$ and extend it to a sheaf $\mathcal{A} = j_*\mathcal{A}_U$ on $\mathbb{P}^1$, where $j : U \hookrightarrow \mathbb{P}^1$ is the open immersion. The conjugacy class of the local monodromy around each of the singular points is a unipotent Jordan block. In fact, the monodromy representation is well-known and can be derived from the general Gamma conjecture, whose relationship to monodromy is discussed in [24] and [48]. The corresponding monodromy group turns out to be $\Gamma_1(5)$, where the 5 ultimately stems from the degree of $G(2,5)$; see [23] where the gamma conjecture is proved for grassmannians.

The modular origin of this operator was discovered long ago by F. Beukers [6]: the Apéry operator is the Picard–Fuchs operator of a rational elliptic surface

$$\pi : E_A \longrightarrow \mathbb{P}^1,$$

which can be identified, after choice of a section, with the universal elliptic curve over the modular curve $X_1(5) = \mathbb{P}^1$. We call this the Apéry family. The fibers of π are elliptic curves with a 5–torsion point; over the four cusps of $X_1(5)$ we find generalized elliptic curves of Kodaira type I_5, I_1, I_1, I_5.

The Apéry family has a nice Laurent polynomial description: the Laurent polynomial

$$a(x,y) = \frac{(1+x)(1+y)(1+x+y)}{xy} \in \mathbb{Z}[x, x^{-1}, y, y^{-1}]$$

has the property that the constant term of its n–th power is the Apéry number A_n:

$$A_n = [a(x,y)^n]_0.$$

Consequently, the period function $A(t)=\sum_{n=1}^{\infty} A_n t^n$ can be represented as

$$A(t)=\frac{1}{(2\pi i)^2}\oint\oint \frac{1}{1-ta(x,y)}\frac{dx}{x}\frac{dy}{y}.$$

The polar locus defines a family of open elliptic curves

$$E^{\circ}_{A,t} := \{(x,y)\in \mathbb{G}_m^2 \quad \mid 1-ta(x,y)=0\} \subset \mathbb{G}_m^2,$$

which, by adding five points, compactify to the elliptic curve $E_{A,t}$ of the Apéry family, of which $A(t)$ is a period.

As the family is defined over $\mathbb{Z}$, we can look at its reduction, and the reduction of any curve E_{A,t_0} in it with $t_0 \in \mathbb{Q}$ modulo any prime number p. If the reduction of $E_{A,t_0} \mod p$ is smooth, its number of $\mathbb{F}_p$–points is given by

$$\#E_{A,t_0}(\mathbb{F}_p)=1-a_p(t_0)+p, \quad a_p(t_0)=\operatorname{Tr}\operatorname{Frob}_p,$$

where Frob_p is the Frobenius acting on $H^1_{\text{ét}}\,(E_{A,t_0} \mod p \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$. In practical terms, the consideration of the Hasse invariant shows that

$$a_p(t_0)=A_p(t_0):=\sum_{n=0}^{p-1} A_n t_0^n \qquad (\text{mod } p) \in (-p/2,p/2],$$

so it can be computed in the smooth cases immediately from the period function. As all elliptic curves are modular, the $a_p(t_0)$ are Fourier coefficients of weight 2 newforms $\in S_2(\Gamma_0(N_{t_0}))$, where N_{t_0} is the conductor. This number is determined by the primes of bad reduction of E_{A,t_0}, which one expects to be those that occur in either t_0 or $\Delta_A(t_0)$. For example, for $t_0=1$ one has $\Delta_A(1)=-11$, so one may expect that the fiber $E_{A,1}$ has only bad reduction at 11.

The first entry in the Antwerp table is the curve 11A, sometimes called the first elliptic curve found in nature. It received the code 11.a3 in the LMFDB and can be given by the equation $y^2+y=x^3-x^2$. In fact, the curve $11A$ is just an incarnation of the modular curve $X_1(11)$ and has a 5–isogeny, represented by the map $X_1(11)\longrightarrow X_0(11)$. It is an entertaining exercise to show that the curve defined by the Laurent polynomial $a(x,y)=1$ is indeed the curve 11A.

Due to the presence of a 5–torsion point, the number of points of each of the curves E_{A,t_0} is divisible by 5, so one has

$$1-a_p(t_0)+p\equiv 0 \qquad (\text{mod } 5),$$

which represents a common property of all the modular forms attached to the smooth fibers of the family.

It is convenient to express this property in terms of the Euler factors

$$Q_p(t_0, T) = \det(1 - T \cdot \mathrm{Frob}_p) = 1 - a_p(t_0)T + pT^2$$

by saying

$$Q_p(t_0, T) \equiv (1 - T)(1 - pT) \pmod{5}.$$

2.3. A mirror pencil for $G_{2,2,1}$

For the Calabi–Yau threefold $G_{2,2,1}$ one may compute the characteristic numbers

$$H^3 = 20, \quad c_2 \cdot H = 68, \quad c_3 = -120,$$

where H is the hyperplane class h restricted to $G_{2,2,1}$ and c_2 and c_3 are the respective Chern classes of $G_{2,2,1}$. By the Lefschetz hyperplane theorem, $h^{1,1}(G_{2,2,1}) = 1$, and we find $h^{2,1}(G_{2,2,1}) = 61$, since the Euler characteristic of a threefold equals its c_3. In [4], a mirror dual pencil of Calabi–Yau threefolds with Hodge numbers $h^{1,1} = 61$ and $h^{1,2} = 1$ was described whose Picard–Fuchs operator

$$\mathcal{P}_C = \theta^4 - 4t(2\theta+1)^2(11\theta^2+11\theta+3) - 16t^2(2\theta+1)^2(2\theta+3)^2$$

has been introduced above. The set of singularities Σ_C consists of the four points 0 and ∞ and the roots

$$c_1 = \frac{-11 - 5\sqrt{5}}{32} = a_1/16, \quad c_2 = \frac{-11 + 5\sqrt{5}}{32} = a_2/16$$

of the polynomial

$$\Delta_C(t) = 1 - 176t - 256t^2 = \Delta_A(16t).$$

The local system of solutions $\mathcal{C}_{U'}$ on $U' := \mathbb{P}^1 \setminus \Sigma_C$ has a point of maximal unipotent monodromy at 0 (MUM), two conifold points with Jordan block of size two and eigenvalues 1, i.e. symplectic reflections, and at ∞ two size two Jordan blocks with eigenvalue -1. We put $\mathcal{C} = j'_*\mathcal{C}_{U'}$ where $j' : U' \hookrightarrow \mathbb{P}^1$ is the respective open immersion.

In fact, the complete integral monodromy representation can be given, and in a suitable integral basis its image is contained in a subgroup of matrices in

$\mathrm{Sp}_4(\mathbb{Z})$ whose reduction mod 5 is block–triangular, a 'non-modular analogue' of a congruence subgroup.

The period function of $\mathcal{P}_C$ is

$$C(t) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 A_n t^n = 1 + 12t + 684t^2 + \ldots,$$

where as before the A_n's are the small Apéry numbers. Thus, it can be considered as an Hadamard product of the Legendre period

$$L(t) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 t^n = {}_2F_1(1/2, 1/2, 1; 16t)$$

with the Apéry period

$$A(t) = \sum_{n=0}^{\infty} A_n t^n.$$

The series $L(t)$ can be interpreted as a period for the standard Legendre family, which has Kodaira fibers I_2, I_2, I_2^* and satisfies the hypergeometric Picard–Fuchs operator

$$\mathcal{P}_L := \theta^2 - 16t(\theta + \frac{1}{2})^2.$$

But it can also be considered as a period for the two isogenous elliptic surfaces with Kodaira fibers I_1, I_1, I_4^*, and I_4, I_1, I_1^*. For the Laurent polynomial

$$l(x,y) := \frac{(1+x)^2(1+y)^2}{xy} \in \mathbb{Z}[x, x^{-1}, y, y^{-1}]$$

one has

$$[l(x,y)^n]_0 = \binom{2n}{n}^2$$

and the curves

$$E^{\circ}_{L,t} := \{(x,y) \in \mathbb{G}_m^2 \mid 1 - tl(x,y) = 0\} \subset \mathbb{G}_m^2$$

are compactified to the fibers of the rational elliptic surface $E_L \longrightarrow \mathbb{P}^2$ with Kodaira fibers I_4 at 0, I_1 at $1/16$ and I_1^* at ∞, which is, after a choice of zero–section, isomorphic to the universal elliptic curve with a 4–torsion point.

The above fourth order operator $\mathcal{P}_C$ is said to be the Hadamard product of the Legendre and Apéry operators:

$$\mathcal{P}_C = \mathcal{P}_L *_{\mathbb{G}_m} \mathcal{P}_A.$$

Note that the operation of taking Hadamard product multiplies the singularities of the factors, which explains the factor 16 in going from $\Delta_A(t)$ to $\Delta_C(t)$.

On the level of local systems of solutions, the Hadamard product corresponds to the operation of taking multiplicative convolution:

$$\mathcal{C} = \mathcal{L} *_{\mathbb{G}_m} \mathcal{A}.$$

The operation of convolution of ℓ–adic sheaves on $\mathbb{P}^1$ is described in detail in [35] and has been refined by the work of M. Dettweiler and S. Reiter [16]. In particular, one can give the monodromy representation of the convolution in terms of the monodromy representation of the convolvees.

We can use the convolution structure to describe a corresponding geometrical model for the mirror family of $G_{2,2,1}$. To describe it, we start with the rational elliptic surfaces $E_A \longrightarrow \mathbb{P}^1$ of the Apéry family with I_5, I_1, I_1, I_5 fibers and the surface $E_L \longrightarrow \mathbb{P}^1$ of the Legendre family with Kodaira fibers I_4, I_1, I_1^*.

Definition 2.4. For $t \neq 0$, let $i_t : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be the map $u \mapsto t/u$. If $E \longrightarrow \mathbb{P}^1$ and $F \longrightarrow \mathbb{P}^1$ are two families of varieties over $\mathbb{P}^1$, we call

$$E \times_t F := E \times_{\mathbb{P}^1} i_t^*(F)$$

the t–twisted fibered product of E and F.

In particular, for our two elliptic surfaces E_L and E_A over $\mathbb{P}^1$ we write

$$W_t := E_L \times_t E_A.$$

These varieties come with a canonical morphism $\pi_t : W_t \longrightarrow \mathbb{P}^1$, which provides them with a fibered structure.

Theorem 2.5. *There exists a flat projective family $f : \mathcal{Y} \longrightarrow \mathbb{P}^1$ defined over $\mathbb{Z}$, whose fibres $Y_t := f^{-1}(t)$ for $t \in \mathbb{P}^1 \setminus \Sigma_C$ have the following properties:*

i) there is a crepant resolution morphism $\rho_t : Y_t \longrightarrow W_t$, which means that the canonical class of Y_t is trivial; in fact, *ii) Y_t is a smooth Calabi–Yau threefold with*

$$h^{1,1}(Y_t) = 61, \quad h^{1,2}(Y_t) = 1, \quad \chi(Y_t) = 120,$$

iii) Y_t is a compactification of

$$U_t := \{(x, y, u, v) \in \mathbb{G}_m^4 \mid 1 - ta(x, y)l(u, v) = 0\};$$

iv) the series $\sum_{n=0}^{\infty}[(l(x,y)a(u,v))^n]_0 t^n$ represents a (normalized) period of $\mathcal{Y} \longrightarrow \mathbb{P}^1$;

v) for each t the group $H^2(Y_t)$ is spanned by divisors defined over $\mathbb{Q}$.

Proof. Step 1: The multiplication map $\mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1, (u, v) \mapsto u \cdot v$ defines a rational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ which extends to a morphism $\mu : \widetilde{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \mathbb{P}^1$ by blowing up the two points of indeterminacy $(0, \infty)$ and $(\infty, 0)$. There are also the two canonical projections $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$. When we pull back $E_L \longrightarrow \mathbb{P}^1$ over the first and $E_A \longrightarrow \mathbb{P}^1$ over the second projection, we obtain two families of elliptic curves over $\mathbb{P}^1 \times \mathbb{P}^1$, of which we can take the fiber product. We pull this back over the blow–up map $\widetilde{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and call the resulting space $\mathcal{W}$. We denote by

$$g : \mathcal{W} \longrightarrow \mathbb{P}^1$$

the composition of the two maps

$$\mathcal{W} \longrightarrow \widetilde{\mathbb{P}^1 \times \mathbb{P}^1} \xrightarrow{\mu} \mathbb{P}^1.$$

Step 2: Untangling the definitions, we see that the fiber of g over a general point t is isomorphic restriction of the fiber product of the families to the hyperbola $u \cdot v = t$ in $\mathbb{P}^1 \times \mathbb{P}^1$, where we use u as coordinate for the first and v for the second copy of $\mathbb{P}^1$. When we parameterize this hyperbola with the first coordinate u, we see that the fiber W_t of $\mathcal{W}$ over t can be identified with the t–twisted fiber product of the family E_L and E_A.

In general, the fiber product of two elliptic surfaces over $\mathbb{P}^1$ is a smooth threefold if the sets of values of the singular fibers are disjoint. If the two surfaces are (relatively minimal) rational elliptic surfaces, Schoen [46] has shown that the resulting threefold is a Calabi–Yau threefold. If singular fibers of the two surfaces 'collide', the threefold in general becomes singular, but depending on the case, there may or may not exist a crepant resolution.

Step 3: In the case at hand we have, for $t \in \mathbb{P}^1 \setminus \Sigma_C$, the following relative position of the singular fibres of E_L and E_A.

0	t/a_1	$1/16$	t/a_2	∞
I_4	$-$	I_1	$-$	I_1^*
I_5	I_1	$-$	I_1	I_5

(Here – denotes a smooth fibre). Over $u = 0$ the fiber of the map $W_t \longrightarrow \mathbb{P}^1$ is the cartesian product of the I_4 and I_5 fibers. So, it consists of $20 = 4 \cdot 5$ divisors isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The threefold W_t has $20 = 4 \cdot 5$ isolated singularities of type A_1 corresponding to the cartesian product of the nodes of the I_4 and I_5 fibers of the E_L and E_A. Over $u = \infty$ the geometry is a bit more involved: it consists of the cartesian product of the I_1^* fiber of E_L and the other I_5 fiber of E_A. The I_1^* fiber consists of $6\,\mathbb{P}^1$'s that intersect in the pattern of the $\widetilde{D_5}$–graph. So above $u = \infty$ we find $30 = 6 \cdot 5$ divisors on the threefold W_t. However, the I_1^* fiber is non-reduced; the two middle curves appear with multiplicity two. As a result, the threefold W_t is singular along $2 \cdot 5 = 10\,\mathbb{P}^1$'s, the cartesian product of the 2 non-reduced lines and the nodes of the I_5 fiber. The transversal type of the singularity in the threefold W_t is of type A_1.

The 20 A_1 singularities above $u = 0$ admit a crepant resolution which replaces each singular point by a $\mathbb{P}^1$. By blowing up the $2 \cdot 5$ singular lines that appear over $u = \infty$ in any order, we also obtain a crepant resolution. The space obtained is a smooth Calabi–Yau threefold that we denote Y_t. In this way we obtain a crepant resolution ρ_t from Y_t to W_t. The blow–up centers used do not depend on t, so doing this across the entire family we obtain a projective family $\mathcal{Y} \longrightarrow \mathbb{P}^1$, with generic fiber Y_t.

Step 4: We compute the Euler number of Y_t (for $t \in \mathbb{P}^1 \setminus \Sigma_C$) using the fibration $\pi = \pi_t \rho_t : Y_t \longrightarrow \mathbb{P}^1$. By the additivity of the Euler number and the fact that for $u \neq 0, \infty$ the fiber over u has Euler number 0 (because the fiber contains at least one elliptic curve factor) we see that $\chi(Y_t)$ is the sum of the contributions over 0 and ∞. Using the geometry, we find the contribution from 0 to be $2 \cdot 4 \cdot 5 = 40$; above ∞ we have $5 \cdot 16$, so that in total we find $\chi(Y_t) = 40 + 80 = 120$.

Step 5: In order to describe the Picard group of Y_t, we follow an idea used already in [46]. Let V be the subgroup of $\mathrm{Pic}(Y_t)$ generated by all vertical divisors of $\pi : Y_t \longrightarrow \mathbb{P}^1$, i.e. generated by components of the fibers. All relations between these divisors are induced from the linear equivalence of the fibers. Over 0 we have $4 \cdot 5 = 20$ divisors, over ∞ we have $5 \cdot 6 + 5 \cdot 2 = 40$

divisors. So we have

$$\operatorname{Rank} V = (20-1) + (40-1) + 1 = 59.$$

The quotient of $\operatorname{Pic}(Y_t)$ by V can be identified with $\operatorname{Pic}(Y_{t,\eta})$, the Picard group of the scheme theoretic generic fiber of $Y_t \longrightarrow \mathbb{P}^1$, so we have

$$\operatorname{Rank}\operatorname{Pic}(Y_t) = 59 + \operatorname{Rank}\operatorname{Pic}(Y_{t,\eta}).$$

Because the curves $E_{L,\eta}$ and $E_{A,\eta}$ are not isogenous, it follows that $\operatorname{Pic}(Y_{t,\eta}) = \operatorname{Pic}(E_{L,\eta})\times \operatorname{Pic}(E_{A,\eta})$. Furthermore, $\operatorname{Rank}\operatorname{Pic}(E_{L,\eta}) = 10-(3+5+1) = 1$ and $\operatorname{Rank}\operatorname{Pic}(E_{A,\eta}) = 10-(4+4+1) = 1$. We finally obtain

$$h^{1,1} = \operatorname{Rank}\operatorname{Pic}(Y_t) = 59 + 2 = 61.$$

Note that this also shows that we can find generators of $\operatorname{Pic}(Y_t)$ defined over the ground field.

Step 6: As $\chi(Y_t) = 120$ and $h^{1,1} = 61$ we see that indeed $h^{1,2} = 1$. So H^3Y_t is of rank four and of Hodge type $(1,1,1,1)$. The modulus of Y_t is visible: it given by the scaling parameter t.

Step 7: As the Laurent polynomial l and a define Zariski open parts of E_L and E_A respectively, the Laurent polynomial $1-tl(x,y)a(u,v)$ defines an open part of $E_L \times_t E_A$. □

Just as in the Apéry family we obtained the minimal conductor for $t_0 = \pm 1$, we observe that $\Delta_C(-1) = -79$, which suggests that the fiber Y_{-1} at $t_0 = -1$ has good reduction for all primes $p \neq 79$. We denote this variety, by slight abuse of notation, $\mathbf{Y}_{79}$.

We are tempted to ask the

2.6. Question

Weissauer's analogue [49] of Eichler's theorem attaches to the paramodular form F_{79} mentioned in 1.3 a $\operatorname{Gal}(\mathbb{Q})$–representation. Is it isomorphic to the natural representation on $H^3\mathbf{Y}_{79}$? Or, at the level of L–functions, does one have

$$L(H^3\mathbf{Y}_{79}, s) = L(F_{79}, s)\,?$$

Let us look at the evidence. The Dirichlet series $L(F_{79}, s)$ starts as

$$L(F_{79}, s) = \sum_{n=1}^{\infty} \frac{a_n(F_{79})}{n^s} = 1 - \frac{5}{2^s} - \frac{5}{3^s} + \frac{11}{4^s} + \frac{3}{5^s} + \frac{25}{6^s} + \frac{15}{7^s} + \ldots$$

To see if we are on the right track, we begin by thinking of our operator

$$\mathcal{P}_C = \theta^4 - 4t(2\theta+1)^2(11\theta^2+11\theta+3) - 16t^2(2\theta+1)^2(2\theta+3)^2$$

as a global crystal and recovering its Frobenius traces. In practice, one starts with a quick $\bmod p$ check by calculating

$$s_p := \sum_{n=0}^{p-1}(-1)^n \binom{2n}{n}^2 A_n \pmod{p}.$$

The result is

p	2	3	5	7	11	13	17	19
s_p	1	1	3	1	4	11	8	13

We observe that indeed

$$s_p \equiv a_p \pmod{p}.$$

If one wants to invest a bit more energy, one can obtain more information by calculating the beginning of the p–adic expansion of the unit root u_p whose first p–adic digit is s_p. As a p–adic number it can be computed, after Dwork [21], as the limit of truncations of the period integrals:

$$\lim_{s\to\infty} \frac{\phi_0^{(s)}(-1)}{\phi_0^{(s-1)}(-1)} \in \mathbb{Z}_p$$

where for a fixed prime p we set

$$\phi_0^{(s)}(x) := \sum_{n=0}^{p^s-1}\binom{2n}{n}^2 A_n x^n,$$

which for $s=1$ reduces to the previous sum. One finds without much difficulty:

$$u_2 = 1 + 2^2 + 2^3 + O(2^9)$$
$$u_3 = 1 + 2\cdot 3 + 3^2 + 2\cdot 3^3 + 2\cdot 3^5 + 2\cdot 3^6 + 2\cdot 3^7 + 2\cdot 3^8 + O(3^9)$$
$$u_5 = 3 + 3\cdot 5 + 2\cdot 5^2 + 4\cdot 5^4 + 3\cdot 5^5 + 4\cdot 5^6 + O(5^7)$$
$$u_7 = 1 + 4\cdot 7 + 2\cdot 7^2 + 7^3 + 5\cdot 7^6 + O(7^7)$$

One can verify that these are indeed roots of the corresponding Euler factors up to required precision. This is already good evidence that the Galois representation on $H^3(\mathbf{Y}_{79})$ is related to the one attached to F_{79}.

2.7. Towards a p–adic gamma conjecture: a practical digression

The next level of sophistication after $\bmod p$ and unit–root calculations is to compute the differential equation for the full Frobenius in order to get the traces. The reason why we are discussing it here rather than merely giving the results is that we do it in a slightly unconventional way by following the method in [9]. However, it should be stressed that the correctness of this method is conjectural at the moment, even though one finds a perfect match with the available data.

The calculation, which amounts to computing the matrix of Frobenius to a certain p–adic precision in the pencil $f : \mathcal{Y} \longrightarrow \mathbb{P}^1$, uses the Picard–Fuchs equation and an initial condition for the Frobenius $\mathrm{F}_p(0)$. One writes a power series expansion of the matrix of Frobenius F_p at t as

$$\mathrm{F}_p(t) = E(t^p)^{-1}\ \mathrm{F}_p(0)E(t),$$

where $E(t) \in \mathbb{Q}[[t]]^{4\times 4}$ is the fundamental solution to the differential equation taken with respect to the Frobenius basis of solutions $\{\varpi_j(t)\}$, so that $E_{ij}(t) = \frac{1}{i!}\Theta^i \varpi_j(t)$. The matrix $\mathrm{F}_p(0)$ is the 'Frobenius matrix at 0' that should be interpreted as the Frobenius on the nearby cohomology $H^3_{\lim}(\mathcal{Y})$. Conjecturally it always takes the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ \gamma p^3 & 0 & 0 & p^3 \end{pmatrix}$$

where

$$\gamma = r\zeta_p(3),$$

for some $r \in \mathbb{Q}$ not depending on p. Here $\zeta_p(3)$ is defined for each prime p as a value of the Kubota–Leopoldt L–function: $\zeta_p(3) = L_p(3, \omega^{-2})$. In terms of Morita's gamma function, $\zeta_p(3) = -\frac{1}{2}(\Gamma_p'''(0) - \Gamma_p'(0)^3)$, see [9, Appendix B]. In our case $r = -\frac{\chi(G_{2,2,1})}{h^3} = \frac{120}{20} = 6$.

The claim is roughly that for $\bar{t}_0 \in \mathbb{F}_p^*$ one has

$$\mathrm{F}_p(\tilde{t_0}) = \mathrm{Frob}_p : H^3_{\text{cris}}\ (\bar{X}_{\bar{t}_0}) \longrightarrow H^3_{\text{cris}}\ (\bar{X}_{\bar{t}_0})$$

where $\tilde{t_0}$ is the Teichmüller lift of $\bar{t}_0 \in \mathbb{F}_p^*$. Practically, the calculation involves the following steps. One first computes, once and for all, the series $E(t)$ to sufficiently many terms. A few hundred terms usually suffice to calculate for all primes $p < 100$. Then, given a prime p, we expand the initial segment of the series $\mathrm{F}_p(t) \in \mathbb{Q}[[t]]^{4\times 4}$ to the same precision as $E(t)$. The parameter γ is left to be determined in the series. For almost all primes p, the entries of this rational matrix series are p–adic integers, and we reduce the series $\bmod p^4$. The result is a matrix $\mathrm{F}_p(t) \pmod{p^4} \in (\mathbb{Z}/p^4\mathbb{Z})[[t]]^{4\times 4}$. It appears that the series is in fact a *polynomial of low degree in* $t \bmod p^4$ for a unique choice of $\gamma \bmod p$. The degrees of the series $\bmod p$, $\bmod p^2$, $\bmod p^3$, $\bmod p^4$ are expected to be close to p times the exponents of the operator at ∞. (One can continue by calculating $\bmod p^5$, $\bmod p^6, \ldots$, and the pattern extends as long as we pick further p–adic digits for $\gamma \bmod p^2, \gamma \bmod p^3, \ldots$.) One now evaluates the series $\mathrm{F}_p(t) \bmod p^4$ at the Teichmüller lift $\tilde{t_0}$ of the parameter $\bar{t}_0 \in \mathbb{F}_p^*$ in question. In our case we evaluate just at -1. Take the characteristic polynomial $P(T) \in (\mathbb{Z}/p^4\mathbb{Z})[T]$ of the resulting matrix. We write the Euler factors for the fiber at -1 in the form

$$Q_p(-1,T) = 1 + \alpha_p T + \beta_p p T^2 + \alpha_p p^3 T^3 + p^6 T^4 \in \mathbb{Z}[T].$$

From the conjectural congruence $Q_p(-1,T) \equiv P(T) \pmod{p^4}$ and the Weil bounds one can uniquely determine $Q_p(-1,T)$ from its reduction $\bmod p^4$.

Taking just 200 terms, with this method one computes in a few seconds the putative Euler factors for the fiber $\mathbf{Y}_{79} = Y_{-1}$ at $t_0 = -1$ for the primes $7 \le p \le 103$. The result is

p	α_p	β_p
7	-15	26
11	-26	122
13	15	12
17	60	134
19	-32	-350
23	-50	274

p	α_p	β_p
29	-24	-146
31	-142	322
37	500	4038
41	-240	2558
43	320	3138
47	105	3602

p	α_p	β_p
53	-630	4130
59	-25	158
61	-194	5402
67	-420	8050
71	-111	-4390
73	-460	5398

p	α_p	β_p
79	-127	6289
83	90	5402
89	1261	11404
97	-1895	26604
101	1239	15572
103	2105	27386

These computations are very fast and may be continued without much effort much further. We do, in fact, need this tabular information for a possible future proof of $L(\mathbf{Y}_{79}, s) = L(F_{79}, s)$ by e.g. the Faltings–Serre method, so we now turn to the unconditional methods to obtain the Frobenius traces: direct point count and its elaboration, point count by convoluting trace functions in elliptic surfaces. Recall that we have denoted by U_t the open part of the fibered product,

$$U_t = \{(x, y, u, v) \in \mathbb{G}_m^4 \mid 1 - ta(x, y)l(u, v) = 0\}.$$

Proposition 2.8. *For any prime power q one has*

$$\#U_{-1}(\mathbb{F}_q) = q^3 - 8q^2 + 21q - 23 - \mathrm{Trace}_q(H^3\mathbf{Y}_{79}).$$

Proof. As we know that all divisors are defined over the ground field, it follows the number of points of $\mathbf{Y}_{79}$ is

$$\#\mathbf{Y}_{79}(\mathbb{F}_q) = 1 + 61(q + q^2) + q^3 - \text{ Trace }_q(H^3\mathbf{Y}_{79})$$

Recall that there is a crepant morphism

$$\rho_t : Y_t \longrightarrow E_L \times_t E_A$$

and a morphism of the fiber product $\pi_t : E_L \times_t E_A \longrightarrow \mathbb{P}^1$. From the explicit resolution process we see that part of Y_0 of Y_{79} lying over $0 \in \mathbb{P}^1$ contributes

$$\#Y_0(\mathbb{F}_q) = 5 \times 4 \cdot (q + q^2),$$

whereas the part Y_∞ lying over ∞ contributes

$$\#Y_\infty(\mathbb{F}_q) = 5 \times 8 \cdot (q + q^2).$$

So for the part Y° lying over $\mathbb{G}_m$ we obtain

$$\#Y^\circ(\mathbb{F}_q) = 1 + (q + q^2) + q^3 - \mathrm{Trace}_q(H^3\mathbf{Y}_{79}).$$

As E_L is a rational elliptic surface, the part E_L° over $\mathbb{G}_m$ has $1 + 10q + q^2 - 4q - (1 + 6q) = q^2$ points, where as for the part E_A° over $\mathbb{G}_m$ we find $1+10q+q^2-5q-5q = 1+q^2$ points. For the number of points in the torus $\mathbb{G}_m^4$ we thus find:

$$\begin{aligned}\#U_{-1}(\mathbb{F}_q) &= \#X^\circ(\mathbb{F}_q) - 5E_L^\circ(\mathbb{F}_q) - 4E_A^\circ(\mathbb{F}_q) + 4 \times 5 \times (q - 1)\\ &= \#Y^\circ(\mathbb{F}_q) - 9q^2 + 20q - 24\\ &= q^3 - 8q^2 + 21q - 23 - \text{ Trace}_q(H^3\mathbf{Y}_{79}).\end{aligned}$$

□

Using this result one can verify by a direct point count in $\mathbb{G}_m^4$ the equality $\mathrm{Trace}_p(H^3\mathbf{Y}_{79}) = a_p(F_{79})$ for small p.

So there is good evidence that the Galois representation on $H^3(\mathbf{Y}_{79})$ might be realizing the Galois representation attached to F_{79}; we can extend this to higher values of p and to points over $\mathbb{F}_{p^2}$ to obtain the complete Euler factors at p.

Alternatively, in order to judge the correctness of the Euler factors obtained by the crystalline method, we could check the functional equation. It is convenient to use the functionality provided by MAGMA. Taking the first 397 terms of the Dirichlet series one gets the functional equation correct up to 30 digits.

Finally, we refer the reader to [20], where the Euler factors for F_{79} are computed with a Brandt–module technique for O(5)–orthogonal forms.

At the moment we cannot prove that the Galois representations attached to F_{79} and $H^3\mathbf{Y}_{79}$ are isomorphic. Although an effective Faltings–Serre strategy for GSp_4 was developed and successfully used in [7], this requires the irreducibility of the residual representation at the prime 2. But the residual representation of F_{79} is reducible, basically due to the 2 congruence implied by the 32 congruence mentioned in the introduction but not treated in this paper.

2.9. Fibering out H^3Y_t

Recall that we had for $t \in \mathbb{P}^1 \setminus \Sigma_C$ the representation of the Calabi–Yau manifold Y_t as a crepant resolution of the twisted fiber product

$$W_t = E_L \times_t E_A$$

In fact there are different choices for this crepant resolution, so there are different spaces Y_t which all are birational to each other. As we are only interested in the motive H^3Y_t, the differences between these different crepant birational models are unimportant to us; all these H^3's are isomorphic to the pure weight three part $Gr_3^W H^3(W_t)$.

One can express these objects directly in terms of the $*$–extended local systems $\mathcal{L}$ and $\mathcal{A}$. We will use the shorthand notation for the t–twisted tensor product:

$$\mathcal{L} \otimes_t \mathcal{A} := j_*(\mathcal{L}_U \otimes i_t^* \mathcal{A}_U),$$

where in the RHS, we have in turn agreed to abuse notation so that j_* is an umbrella sign meaning 'star–extension of anything from a dense open subset to $\mathbb{P}^1$, and $\otimes$ means the tensor product of two local systems on a dense open subset where both are defined, much the same way they are used in spoken mathematics. To aggravate things further, we use the same letters $\mathcal{L}$ and $\mathcal{A}$ for the pullbacks of the respective coefficient systems under the base change to algebraic closure. By $i_t : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ we mean as before the map $u \mapsto t/u$.

Theorem 2.10. *For $t_0 \in (\mathbb{P}^1 \setminus \Sigma_C)(\mathbb{Q})$ one has an isomorphism*

$$H^3_{ét}(\bar{Y}_{t_0}, \mathbb{Q}_l) = H^1(\bar{\mathbb{P}}^1, \mathcal{L} \otimes_{t_0} \mathcal{A})$$

of pure weight 3 Galois representations.

Proof. We consider the fibration $\pi : \bar{Y}_{t_0} \longrightarrow \bar{\mathbb{P}}^1$, and let

$$\bar{U} = \bar{\mathbb{P}}^1 \backslash \bar{S},\ S = \{0, \frac{t_0}{a_1}, \frac{1}{16}, \frac{t_0}{a_1}, \infty\}$$

be the part over which π is smooth. The direct image sheaf $R^2\pi_*\mathbb{Q}_l$ restricted to $\bar{U}$ is isomorphic to $\mathcal{L}_{\bar{U}} \otimes_{t_0} \mathcal{A}_{\bar{U}}$. By the local invariant cycle theorem [42], sections over the punctured neighborhood of $R^2\pi_*\mathbb{Q}_l$ extend over the puncture. By the definition of the $*$–extension this means that $R^2\pi_*\mathbb{Q}_l$ surjects onto $\mathcal{L} \otimes_{t_0} \mathcal{A}$, and the kernel is a punctual sheaf. So from the cohomology sequence one gets the isomorphism

$$H^1(\bar{\mathbb{P}}^1, R^2\pi_*\mathbb{Q}_l) = H^1(\bar{\mathbb{P}}^1, \mathcal{L} \otimes_{t_0} \mathcal{A}).$$

From the Leray spectral sequence $E_2^{p,q} = H^p(\bar{\mathbb{P}}^1, R^q\pi_*\mathbb{Q}_l) \Longrightarrow H^{p+q}(\bar{Y}_{t_0}, \mathbb{Q}_l)$ we see that there are no non-trivial arrows going from or to $H^1(\bar{\mathbb{P}}^1, R^2\pi_*\mathbb{Q}_l)$, so this term injects into the E_∞–term of the spectral sequence, hence it is a subquotient of $H^3_{\text{ét}}(\bar{Y}_{t_0}, \mathbb{Q}_l)$. It suffices to prove it has dimension 4. By Euler–Poincaré one has for any sheaf $\mathcal{M}$ on $\bar{\mathbb{P}}^1$ that is $*$–extended from a local system on $\bar{U} \overset{j}{\hookrightarrow} \bar{\mathbb{P}}^1$

$$\chi(\mathcal{M}) = \operatorname{Rank} \mathcal{M} \cdot \chi(\bar{U}) + \sum_{s \in \bar{S}} \dim \mathcal{M}^{I(s)} = 2 \cdot \operatorname{Rank} \mathcal{M} - \sum_{s \in \bar{S}} R(s),$$

where

$$R(s) := \dim \mathcal{M}/\mathcal{M}^{I(s)}$$

denotes the ramification, or drop of $\mathcal{M}$ at s. If $\mathcal{M}$ is self–dual and $h^0(\mathcal{M}) = 0$, this simplifies to the useful formula

$$h^1(\mathcal{M}) = \sum_s R(s) - 2 \cdot \operatorname{Rank} \mathcal{M}.$$

For the sheaf $\mathcal{L} \otimes_t \mathcal{A}$ we have at 0 the tensor product of two unipotent 2×2 matrices, hence a drop of 2. At the singularities of $\mathcal{L}$ and $\mathcal{A}$ on $\bar{\mathbb{G}}_m$ we get the tensor product of identity with a unipotent, so again a drop of 2. At ∞ we have the tensor product of a unipotent matrix and a $(-1) \otimes$ a unipotent 2×2 matrix, which contributes the drop of 4. From this we get $h^1(\mathcal{L} \otimes_t \mathcal{A}) = 4$. □

Corollary 2.11. *One has the following formula for the trace function*

$$\mathrm{Trace}_p(t_0) := \mathrm{Tr}\,\mathrm{Frob}_p,\ \mathrm{Frob}_p : H^3_{ét}\,(\bar{Y}_{t_0}, \mathbb{Q}_l) \longrightarrow H^3_{ét}\,(\bar{Y}_{t_0}, \mathbb{Q}_l)$$

as the convolution of the corresponding trace functions l_p and a_p of the l–adic sheaves $\mathcal{L}$ and $\mathcal{A}$ (restricted to $\mathbb{P}^1$ over $\mathbb{F}_p$):

$$\mathrm{Trace}_p(t_0) = -\left(1 + p + \sum_{u \in \mathbf{G}_m(\mathbb{F}_p)} l_p(t_0/u) a_p(u)\right).$$

□

One can think of the functions l_p and a_p as giving the traces of Frobenius arising in the Néron models of the elliptic curves producing $\mathcal{L}$ and $\mathcal{A}$ over $\mathbb{F}_p(t)$. We remark here that this simple relationship follows directly from the formalism of ℓ–adic sheaves. A cumbersome proof could be given running the point counting formula relating counts on Y_{t_0} and counts on U_{t_0} involving the correction term. As we are now in a pure situation, all polynomials in p must cancel.

Theorem 2.12. *There exists a 5–congruence for $H^3 Y_{t_0}$.*

$$\mathrm{Trace}_p(t_0) \equiv \sum_{u \in (\mathbf{G}_m \cap \Sigma_A)(\mathbb{F}_p)} l_p(t_0/u) \pmod 5 :$$

Proof. Let us first assume that $p = 2$ or $3 \mod 5$ so that the roots of $\Delta_A(t) = 1 - 11t - t^2$ are defined over $\mathbb{F}_{p^2}$ We recall from (1) that $a_p(u) \equiv 1 + p \pmod 5$ for all $u \in (\mathbb{P}^1 \setminus \Sigma_A)(\mathbb{F}_p)$. This gives

$$\mathrm{Trace}_p(t_0) \equiv -(1+p)\left(1 + \sum_{u \in \mathbf{G}_m(\mathbb{F}_p)} l_p(t_0/u)\right) \pmod 5.$$

Since we know that $h^1(\bar{\mathbb{P}}^1, j_*\mathcal{L}) = 0$, the sum of the traces $1 + \sum_{u \in \mathbf{G}_m(\mathbb{F}_p)} l_p(t_0/u)$ must be zero.

Assume now that $p = 1 \mod 5$. The curve producing $\mathcal{A}$ has split multiplicative reduction at points in $\Sigma_A(\mathbb{F}_p)$, which introduces a correction term there:

$$\mathrm{Trace}_p(t_0) \equiv -(1+p)\left(1 + \sum_{u \in \mathbf{G}_m(\mathbb{F}_p)} l_p(t_0/u)\right) + p \sum_{u \in \text{ roots of } \Delta_A} l_p(t_0/u) \pmod 5.$$

Similarly, for $p = 4 \mod 5$ the curve producing $\mathcal{A}$ has non-split multiplicative reduction at the two roots of $\Delta_A(t) = 1 - 11t - t^2$, and

$$\begin{aligned}\mathrm{Trace}_p(t_0) \equiv -(1+p)\left(1 + \sum_{u \in \mathbf{G}_m(\mathbb{F}_p)} l_p(t_0/u)\right) + \\ (p+2) \sum_{u \in \text{ roots of } \Delta_A} l_p(t_0/u) \pmod 5.\end{aligned}$$

In all cases,

$$\mathrm{Trace}_p(t_0) \equiv \sum_{u \in (\mathbf{G}_m \cap \Sigma_A)(\mathbb{F}_p)} l_p(t_0/u) \qquad \pmod 5.$$

□

Thus the Euler factors of the L-function at t_0 we are interested in are determined mod 5 by the point count (over $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$) on the two fibers E' and E'' of the Legendre family over the two points determined by the two singularities of the Apéry family. These two Galois conjugate curves are defined over $\mathbb{Q}(\sqrt{5})$, and thus are attached by [22] to two (conjugate) weight $(2,2)$–Hilbert modular forms f_{79}, f'_{79}. For the rank four L–functions one has $L(E', s) = L(E'', s) = L(f_{79}, s) = L(f'_{79}, s)$, hence

$$Q_p(H^3 Y_{t_0}, T) \equiv Q_p(E', T) \equiv Q_p(E'', T) \equiv Q_p(f_{79}, T) \equiv Q_p(f'_{79}, T) \pmod 5.$$

In the special case of $H^3\mathbf{Y}_{79}$, the curves E' and E'' can be found in LMFDB as 79.1-a1 and 79.2-a2, to which Hilbert modular forms 2.2.5.1-79.1-a and 2.2.5.1-79.1-b are attached and which we called just f_{79} in the introduction.

The corresponding Euler factors of the rank four L–function of conductor $1975 = 5^2 79$ are given in LMFDB under the code 4-1975-1.1-c1e2-0-0 and have the Euler factors

p	Q_p
2	$1 - T^2 + p^2T^4$
3	$(1 - 2T + pT^2)(1 + 2T + pT^2)$
5	$1 + 2T + pT^2$
7	$1 + 2T^2 + p^2T^4$
11	$(1 + pT^2)(1 + 4T + pT^2)$
13	$1 + 6T^2 + p^2T^4$
17	$(1 - 6T + pT^2)(1 + 6T + pT^2)$
19	$(1 - 8T + pT^2)(1 - 4T + pT^2)$
23	$(1 - 8T + pT^2)(1 + 8T + pT^2)$

For primes split in $\mathbb{Q}(\sqrt{5})$ the factorization is manifest over $\mathbb{Q}$ and reduces mod 5 to the factorizations seen for $H^3\mathbf{Y}_{79}$ in 1.5; for the inert primes the trace vanishes and the Euler factor is a polynomial in T^2 and may or may not factor over $\mathbb{Q}$, but its factorization mod 5 is elementary.

—

According to Johnson–Leung and Roberts [34], there is a lift of Hilbert modular forms of non-parallel weight $(2, 4)$ to weight 3 paramodular forms. Gonzalo Tornaría has shown that there exists a $(2, 4)$–Hilbert modular form h_{79} whose lift is congruent mod 5 to F_{79}. We have shown that the Dirichlet series of our weight $(2, 2)$–Hilbert modular form f_{79} is congruent mod 5 to the Dirichlet series of $H^3\mathbf{Y}_{79}$ and, conjecturally, with that of F_{79}. But the mod 5 congruence between h_{79} and f_{79} is at present only observed experimentally, again by Tornaría.

Weight $(2, 4)$–Hilbert modular forms are much closer to the motives attached to threefolds. The two Galois conjugates together define a four–dimensional $\mathrm{Gal}(\mathbb{Q})$ representation. There exist some beautiful examples of Calabi–Yau incarnations for such $(2, 4)$–Hilbert modular forms. The Consani–Scholten quintic [13, 18] is a Calabi–Yau threefold X defined over $\mathbb{Q}$ for which the $\mathrm{Gal}(\mathbb{Q})$ representation on H^3X is of this type. It splits when restricted to $\mathrm{Gal}(\mathbb{Q}(\sqrt{5}))$ into two conjugate representations H, H' attached to a weight $(2, 4)$–Hilbert modular form of level $6\sqrt{5}$, so

$$H^3X = \mathrm{Ind}_{\mathrm{Gal}(\mathbb{Q}(\sqrt{5}))}^{\mathrm{Gal}(\mathbb{Q})} H = \mathrm{Ind}_{\mathrm{Gal}(\mathbb{Q}(\sqrt{5}))}^{\mathrm{Gal}(\mathbb{Q})} H'.$$

A similar example was found in [14] for the field $\mathbb{Q}(\sqrt{2})$. In this case the splitting was shown to be due to the presence of an explicit endomorphism defined over the field $\mathbb{Q}(\sqrt{2})$.

Returning to the situation of this paper, one might speculate about the existence of another Calabi–Yau variety Z_{1975} defined over $\mathbb{Q}$, whose H^3 splits into two conjugate representations attached to h_{79} when restricted to $\mathbb{Q}(\sqrt{5})$, which is 'congruent mod 5' to $\mathbf{Y}_{79}$.

Of course, there are many further examples of convolutions to be studied. For instance, the Calabi–Yau section $G_{3,1,1}$ of $G(2,5)$ gives rise to the period function

$$D(t) = \sum_{n=0}^{\infty} \frac{3n!}{n!^3} A_n t^n = H *_{\mathbb{G}_m} A(t)$$

that is the convolution of the period

$$H(t) = \sum_{n=0}^{\infty} \frac{3n!}{n!^3} t^n$$

of an elliptic surface with Kodaira fibers I_3, I_1, IV^* with the Apéry period $A(t)$. In a similar way we get a family of Calabi–Yau threefolds $\mathcal{Z} \longrightarrow \mathbb{P}^1$ with fibers Z_t having $h^{1,1} = 74, h^{1,2} = 1$. The Galois representation on $H^3 Z_t$ will now have a 5– and a 3–congruence of Type III. For $t = -1$ we seem to obtain a motive with the same Euler factors as the paramodular form $431.b$ in [44].

Acknowledgements

Don Zagier has been a source of inspiration for us, contributing greatly to the subjects, or rather themes, of this paper — topology, Hilbert modular forms and differential equations of mirror symmetry. We would like to thank him for the support and encouragement over the years we know him.

We would like to thank the members of the International Groupe de Travail on differential equations in Paris and the guests of the Groupe's seminar, specifically Neil Dummigan, Ariel Pacetti, Cris Poor, Gonzalo Tornaría, and John Voight for the insights they have shared with us, and for corrections. We thank Sławek Cynk for helpful discussion on the geometry of the Calabi–Yau threefolds mentioned in the paper.

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